Pareto Control Barrier Function for Inner Safe Set Maximization Under Input Constraint

Xiaoyang Cao, Zhe Fu
University of California Berkeley

Supervisor(s):

Alexandre Bayen

Table of Contents

Problem Formulation

1 Background

Consider a nonlinear, control-affine dynamical system described by:

$$\dot{x} = f(x) + g(x)u \tag{1}$$

with state $x \in \mathcal{X} \subseteq \mathbb{R}^n$ and control input $u \in \mathcal{U} \subseteq \mathbb{R}^m$.

Definition: Safe Set

We define a state x as safe, if it lies in a set $\mathcal{S}$

$$\mathcal{S} \triangleq \{x \in \mathcal{X} : h(x) \geq 0\} \tag{2}$$

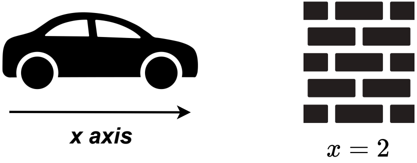
$h : \mathcal{X} \to \mathbb{R}$ is continuously differentiable. $\mathcal{S}$ is referred to as the **safe set**

Definition: Forward Invariant

A set $\mathcal{A}$ remains **forward invariant** under a dynamical system if all trajectories $x(t)$, starting from any $x(0) \in \mathcal{A}$, stay within $\mathcal{A}$ for all $t \geq 0$.

Problem Formulation

- To ensure safety, it's crucial that the trajectory always stays within the safe set $\mathcal{S}$, i.e., $\mathcal{S}$ **is forward invariant.**



- However, input constraints may prevent the safe set from being forward invariant[1]:

$$\dot{x} = x + u, \quad \mathcal{U} = [-1, 1], \quad h(x) = 2 - x$$

At the boundary state $x = 2$, $\dot{h}(2) = -2 - u \leq -1$ for all $u \in \mathcal{U}$, showing that trajectories starting at $x(0) = 2$ will leave the safe set.

[1] Devansh R Agrawal and Dimitra Panagou. "Safe control synthesis via input constrained control barrier functions". In: *2021 60th IEEE Conference on Decision and Control (CDC)*. IEEE. 2021, pp. 6113–6118.

Problem Formulation

Definition: Inner Safe Set

A non-empty closed set $\mathcal{C}$ is an **inner safe set** of the safe set $\mathcal{S}$ for the dynamical system (1) if $\mathcal{C} \subseteq \mathcal{S}$ and there exists a feedback controller $\pi : \mathcal{C} \to \mathcal{U}$ such that $\mathcal{C}$ is rendered forward invariant by π.

Problem: Inner Safe Set Maximization

Determine the largest possible inner safe set $\mathcal{C}^*$ given safe set $\mathcal{S}$ (2) and system dynamics (1).

- **Control Barrier Function (CBF)** offers a methodological framework for designing feedback controllers that ensure the forward invariance of $\mathcal{C}^*$.

Table of Contents

UC Berkeley

Control Barrier Function

Definition: Control Barrier Function

Let $\mathcal{S} = \{x \in \mathcal{X} : h(x) \geq 0\} \subseteq \mathcal{X} \subseteq \mathbb{R}^n$ be a safe set (2), then h is a CBF if there exists an extended class-$\mathcal{K}_\infty$ function α such that:

$$\sup_{u \in \mathcal{U}} [L_f h(x) + L_g h(x) u] \geq -\alpha(h(x)), \quad \forall x \in \mathcal{X} \tag{3}$$

where $L_f h(x) = \nabla h(x)^T f(x)$ and $L_g h(x) = \nabla h(x)^T g(x)$ are the Lie derivatives.

Definition: Extended Class-$\mathcal{K}_\infty$ Function

A function $\alpha : \mathbb{R} \to \mathbb{R}$ is an extended class-$\mathcal{K}_\infty$ function if it is continuous, strictly increasing, unbounded, and $\alpha(0) = 0$.

Lemma 1

[a] Let $\mathcal{S} = \{x \in \mathcal{X} : h(x) \geq 0\} \subseteq \mathcal{X} \subseteq \mathbb{R}^n$ be a safe set (2). If h is a CBF on $\mathcal{X}$, then any Lipschitz continuous controller $\pi(x) \in K_{CBF}(x)$ renders the safe set $\mathcal{S}$ **forward invariant**, where

$$K_{CBF}(x) = \{u \in \mathcal{U} : L_f h(x) + L_g h(x) u \geq -\alpha(h(x))\} \tag{4}$$

[a] Aaron D Ames et al. "Control barrier functions: Theory and applications". In: *2019 18th European control conference (ECC)*. IEEE. 2019, pp. 3420–3431.

Challenges : CBFs do not provide a method to identify inner safe set. Several alternative methods have been proposed to address this issue.

- **HJ** : Hamilton-Jacobi Reachability
- **NCBF** : Neural Control Barrier Function

HJ Reachability

2 Methodology

HJ [2]: Computing the viability kernel $\mathcal{S}(t)$

$$\mathcal{S}(t) \triangleq \{x \in \mathcal{S} : \exists u \in \mathcal{U} \ s.t. \ \forall t' \in [t, 0], x(t') \in \mathcal{S}\}, \quad t < 0 \tag{5}$$

Hamilton-Jacobi-Isaacs Variational Inequality (HJI VI)

$$\min \left(h(x) - V(x,t), D_t V(x,t) + \max_{u \in \mathcal{U}} D_x V(x,t) \cdot f(x,u) \right) = 0, \quad V(x,0) = h(x) \tag{6}$$

$$V_\infty(x) = \lim_{t \to -\infty} V(x,t) \tag{7}$$

$$\mathcal{S}(t) = \{x \in \mathcal{X} : V(x,t) \geq 0\}, \quad \mathcal{C}^* = \{x \in \mathcal{X} : V_\infty(x) \geq 0\} \tag{8}$$

- **Advantages** : Yields the largest possible inner safe set.
- **Disadvantages** : High computational cost. Cannot be applied to high dimensional system.

[2] Jason J Choi et al. "Robust control barrier–value functions for safety-critical control". In: *2021 60th IEEE Conference on Decision and Control (CDC)*. IEEE. 2021, pp. 6814–6821.

Neural Control Barrier Function

NCBF : Learning the closest approximation to the original safe set.

$$h_\theta(x) = h(x) - \delta_\theta(x) \tag{9}$$

$\delta_\theta(x)$ quantifies the deviation between the **original CBF** $h(x)$ and the **NCBF** $h_\theta(x)$.

$$\delta_\theta(x) = (\mathrm{MLP}_\theta(x))^2 \tag{10}$$

$\delta_\theta(x)$ is the squared output of a Multi-Layer Perceptron (MLP) with parameter $\theta \in \mathbb{R}^\Theta$.

$$\mathcal{C}_{h_\theta} \triangleq \{x \in \mathcal{X} : h_\theta(x) \geq 0\} \tag{11}$$

$\mathcal{C}_{h_\theta}$ is the learned **inner safe set**.

- $h_\theta(x) \leq h(x), \quad x \in \mathcal{X}$
- $\mathcal{C}_{h_\theta} \subseteq \mathcal{S}$

$\{x_i\}_{i=1}^{N}$ are N uniformly sampled data points in $\mathcal{C}_{h_\theta}$

- **Sampling Method** :
  - Previous research: sampling on boundary ∂C_{h_θ}[3]; Computationally-expensive and suboptimal.
  - Ours: Sampling in entire C_{h_θ}; Fast and effective.
- **Volume Loss** :

$$\mathcal{L}_{vol} = \frac{1}{N}\sum_{i=1}^{N}\delta_\theta(x_i) \tag{12}$$

- **Feasibility Loss** :

$$\mathcal{L}_{feas} = \frac{1}{N}\max\{0, -\sup_{u\in\mathcal{U}}[L_f h_\theta(x_i) + L_g h_\theta(x_i)u + \alpha(h_\theta(x_i))]\} \tag{13}$$

[3] Simin Liu, Changliu Liu, and John Dolan. "Safe control under input limits with neural control barrier functions". In: *Conference on Robot Learning*. PMLR. 2023, pp. 1970–1980.

Neural Control Barrier Function

- We assume **linear input constraints**, making $\mathcal{U}$ a polyhedron. Thus, the maximum of the affine objective function occurs at the vertices of $\mathcal{U}$

$$\mathcal{L}_{feas} = \frac{1}{N} \max\{0, -\max_{u \in \mathcal{V}(\mathcal{U})} [L_f h_\theta(x_i) + L_g h_\theta(x_i) u + \alpha(h_\theta(x_i))]\} \tag{14}$$

$$\mathcal{V}(u) \triangleq \{u \in \mathcal{U} : u \textit{ is the vertex of } \mathcal{U}\} \tag{15}$$

Problem: Inner Safe Set Maximization

$$\begin{aligned} \min_\theta \quad & \mathcal{L}_{vol} \\ \text{s.t.} \quad & \mathcal{L}_{feas} = 0 \end{aligned} \tag{16}$$

Pareto Multi-task Learning

Competing Objectives : The losses $\mathcal{L}_{feas}$ and $\mathcal{L}_{vol}$ are inherently competing. It is **generally impossible** to achieve both with zero loss simultaneously.

- $\mathcal{L}_{vol} = 0 \Rightarrow C_{h_\theta} = \mathcal{S}$, it typically results in $\mathcal{L}_{feas} > 0$ because there may be no valid CBF on entire safe set.
- $\mathcal{L}_{feas} = 0 \Rightarrow h_\theta$ is a valid CBF. Then $\mathcal{L}_{vol} > 0$ because $C_{h_\theta} \subset \mathcal{S}$.

Pareto Multi-task Learning : Consider a multi-task learning problem with T tasks and loss

$$\mathcal{L}(\theta) = \begin{bmatrix} \mathcal{L}_1(\theta) & \mathcal{L}_2(\theta) & \cdots & \mathcal{L}_T(\theta) \end{bmatrix}^T, \quad \theta \in \mathbb{R}^\Theta \tag{17}$$

Definition: Pareto Optimality

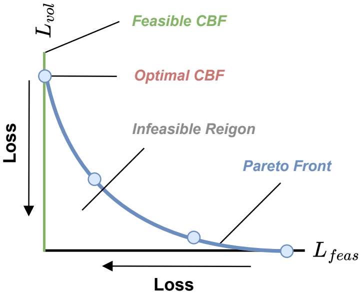
- A solution θ dominates a solution $\bar{\theta}$ if $\forall i, \mathcal{L}_i(\theta) \leq \mathcal{L}_i(\bar{\theta})$ and $\exists i, s.t.\ \mathcal{L}_i(\theta) < \mathcal{L}_i(\bar{\theta})$
- A solution θ^* is called **Pareto Optimal** if there exists no solution θ that dominates θ^*.
- **Pareto Front**: $P_{\mathcal{L}} = \{\mathcal{L}(\theta) \mid \theta \text{ is Pareto optimal}\}$

Pareto Multi-task Learning

Lemma 2

Let θ^* denote the parameter of the optimal NCBF with maximum inner safe set. Then θ^* is pareto optimal and $[\mathcal{L}_{feas}(\theta^*) \quad \mathcal{L}_{vol}(\theta^*)]^T$ is on the pareto front.

- **Previous studies** : Combining two losses through linear combination $\mathcal{L} = \lambda_1 \mathcal{L}_{feas} + \lambda_2 \mathcal{L}_{vol}$. It does not guarantee Pareto Optimality.
- **Our Contribution** : Introducing **Pareto Control Barrier Function** (PCBF), ensuring convergence to optimal parameters on the Pareto front.
- The linear combination method is unable to handle a **concave Pareto front**[4]



[4] Stephen Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge University Press, 2004.

Pareto Control Barrier Function

Pareto Control Barrier Function (PCBF):

- **Initialization**
  Initially, the model is trained using a linear combination loss for a predefined number of iterations.
- **Parameter Update**
  At iteration t, the parameter of the multi-task learning model (17) is θ_t. Then $\theta_{t+1} = \theta_t + \eta d_t$, where $\eta > 0$ is the learning rate, and d_t is obtained as follows:

$$\begin{aligned} (d_t, \alpha_t) &= \arg \min_{d \in \mathbb{R}^{\Theta}, \alpha \in \mathbb{R}} \alpha + \frac{1}{2}\|d\|_2^2 \\ \text{s.t.} \quad & \nabla \mathcal{L}_i(\theta_t)^T d \leq \alpha, \quad \forall i = 1, 2, \cdots, T \end{aligned} \tag{18}$$

Pareto Control Barrier Function

Lemma 3

[a] Let (d_t, α_t) be the solution of problem (18)

- If θ_t is Pareto optimal, then $d_t = 0 \in \mathbb{R}^{\Theta}$ and $\alpha_t = 0$
- If θ_t is not Pareto optimal, then

$$\begin{aligned} \alpha_t &\leq -\frac{1}{2}\|d\|_2^2 < 0 \\ \nabla \mathcal{L}_i(\theta_t)^T d_t &\leq \alpha_t, \forall i = 1, \cdots, T \end{aligned} \tag{19}$$

[a] Jörg Fliege and Benar Fux Svaiter. "Steepest descent methods for multicriteria optimization". In: *Mathematical methods of operations research* 51 (2000), pp. 479–494.

- **Remark**: When $d_t \neq 0$, we have $\nabla \mathcal{L}_i(\theta_t)^T d_t < 0, \forall i = 1, \cdots, T$, which means d_t is a **descent direction for all tasks**.

- **Core Idea of PCBF** : Constrain the loss within the feasible region Ω_β and perform multi-objective optimization within Ω_β, seeking Pareto optimal solutions.

$$\Omega_\beta \triangleq \{\theta \in \mathbb{R}^\Theta : \beta\mathcal{L}_{feas}(\theta) + \epsilon_{lb} \leq \mathcal{L}_{vol}(\theta) \leq \beta\mathcal{L}_{feas}(\theta) + \epsilon_{ub}\}, \quad \beta > 0 \tag{20}$$

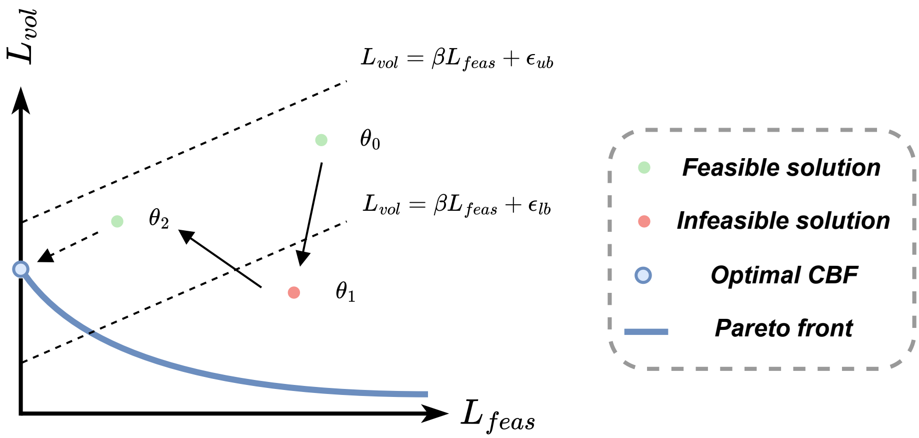
$$0 \leq \epsilon_{lb} \leq \mathcal{L}_{vol}(\theta^*) \leq \epsilon_{ub} \tag{21}$$

Problem:

Find θ satisfying the following conditions:

- $\theta \in \Omega_\beta$.
- $\mathcal{L}(\theta) = [\mathcal{L}_{feas}(\theta), \mathcal{L}_{vol}(\theta)]^T \in P_{\mathcal{L}}$.

Pareto Control Barrier Function



- **Parameter Update** :

$$\theta_{t+1} = \theta_t + \eta d_t \tag{22}$$

where d_t is obtained by solving:

  - If $\theta_t \in \Omega_\beta$:

$$(d_t, \alpha_t) = \arg \min_{d \in \mathbb{R}^\Theta, \alpha \in \mathbb{R}} \alpha + \frac{1}{2}\|d\|_2^2 \tag{23}$$

$$\text{s.t.} \quad \nabla \mathcal{L}_{vol}(\theta_t)^T d \leq \alpha, \tag{24}$$

$$\nabla \mathcal{L}_{feas}(\theta_t)^T d \leq \alpha \tag{25}$$

- If $\theta_t \notin \Omega_\beta$:
  - If $\mathcal{L}_{vol}(\theta) > \beta \mathcal{L}_{feas}(\theta) + \epsilon_{ub}$

$$(d_t, \alpha_t) = \arg \min_{d \in \mathbb{R}^\Theta, \alpha \in \mathbb{R}} \alpha + \frac{1}{2}\|d\|_2^2 \tag{26}$$

$$\text{s.t.} \quad [\nabla(\mathcal{L}_{vol}(\theta_t) - \beta \mathcal{L}_{feas}(\theta_t))]^T d \leq \alpha, \tag{27}$$

$$\nabla \mathcal{L}_{feas}(\theta_t)^T d \leq \alpha \tag{28}$$

  - If $\mathcal{L}_{vol}(\theta) < \beta \mathcal{L}_{feas}(\theta) + \epsilon_{lb}$

$$(d_t, \alpha_t) = \arg \min_{d \in \mathbb{R}^\Theta, \alpha \in \mathbb{R}} \alpha + \frac{1}{2}\|d\|_2^2 \tag{29}$$

$$\text{s.t.} \quad [\nabla(\beta \mathcal{L}_{feas}(\theta_t) - \mathcal{L}_{vol}(\theta_t))]^T d \leq \alpha, \tag{30}$$

$$\nabla \mathcal{L}_{feas}(\theta_t)^T d \leq \alpha \tag{31}$$

The solution of Problem (23), (26) and (29) can be given in a closed form:

- If $\theta_t \in \Omega_\beta$

$$d_t = -\lambda \nabla \mathcal{L}_{vol}(\theta_t) - (1-\lambda)\nabla \mathcal{L}_{feas}(\theta_t) \tag{32}$$

$$\lambda = \max(\min(\frac{(\nabla \mathcal{L}_{feas}(\theta_t) - \mathcal{L}_{vol}(\theta_t))^T \nabla \mathcal{L}_{feas}(\theta_t)}{\|\nabla \mathcal{L}_{vol}(\theta_t) - \nabla \mathcal{L}_{feas}(\theta_t)\|_2^2}, 1), 0) \tag{33}$$

Pareto Control Barrier Function

- If $\theta_t \notin \Omega_\beta$
  - If $\mathcal{L}_{vol}(\theta) > \beta\mathcal{L}_{feas}(\theta) + \epsilon_{ub}$

$$d_t = (\lambda + \lambda\beta - 1)\nabla\mathcal{L}_{feas}(\theta_t) - \lambda\nabla\mathcal{L}_{vol}(\theta_t) \tag{34}$$

$$\lambda = \max(\min(\frac{((1+\beta)\nabla\mathcal{L}_{feas}(\theta_t) - \nabla\mathcal{L}_{vol}(\theta_t))^T\nabla\mathcal{L}_{feas}(\theta_t)}{\|(1+\beta)\nabla\mathcal{L}_{feas}(\theta_t) - \nabla\mathcal{L}_{vol}(\theta_t)\|_2^2}, 1), 0) \tag{35}$$

  - If $\mathcal{L}_{vol}(\theta) < \beta\mathcal{L}_{feas}(\theta) + \epsilon_{lb}$

$$d_t = (\lambda - \lambda\beta - 1)\nabla\mathcal{L}_{feas}(\theta_t) + \lambda\nabla\mathcal{L}_{vol}(\theta_t) \tag{36}$$

$$\lambda = \max(\min(\frac{(\nabla\mathcal{L}_{vol}(\theta_t) + (1-\beta)\nabla\mathcal{L}_{feas}(\theta_t))^T\nabla\mathcal{L}_{feas}(\theta_t)}{\|\nabla\mathcal{L}_{vol}(\theta_t) + (1-\beta)\nabla\mathcal{L}_{feas}(\theta_t)\|_2^2}, 1), 0) \tag{37}$$

Table of Contents

UC Berkeley

- **System Dynamics**

$$\dot{p} = v \tag{38}$$
$$\dot{v} = u \tag{39}$$

- **State Space**

$$\mathcal{X} = \{(p, v) : -6 \leq p \leq 6, -6 \leq v \leq 6\} \tag{40}$$

- **Input Space**

$$\mathcal{U} = \{u : -1 \leq u \leq 1\} \tag{41}$$

- **Safe Set**

$$\mathcal{S} = \{(p, v) : -5 \leq p \leq 5, -5 \leq v \leq 5\} \tag{42}$$

Double Integrator

Loss Type	Sampling Method	Sampling Num	Training Time
Linear Combination	Boundary	10000	5 hours
Linear Combination	Sampling Inside	10000	**20 min**
Pareto	Boundary	10000	5 hours
Pareto	Sampling Inside	10000	**20 min**

Table: Approximate Training Time for Different Sampling Methods

Training conducted on an NVIDIA GeForce RTX 3080 Ti GPU.

Double Integrator Results

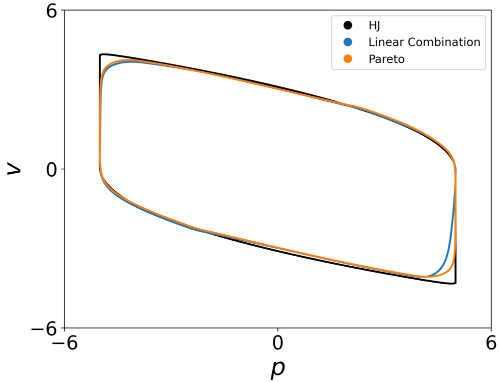


Figure: Sampling Inside

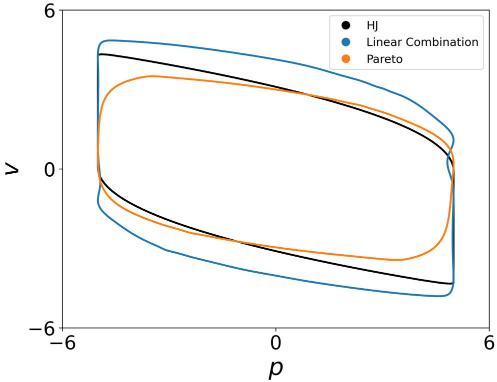


Figure: Sampling on Boundary

- **System Dynamics**

$$\dot{\theta} = \omega \tag{43}$$

$$\dot{\omega} = \sin(\theta) + u \tag{44}$$

- **State Space**

$$\mathcal{X} = \{(\theta, \omega) : -\pi \leq \theta \leq \pi, -1 \leq \omega \leq 1\} \tag{45}$$

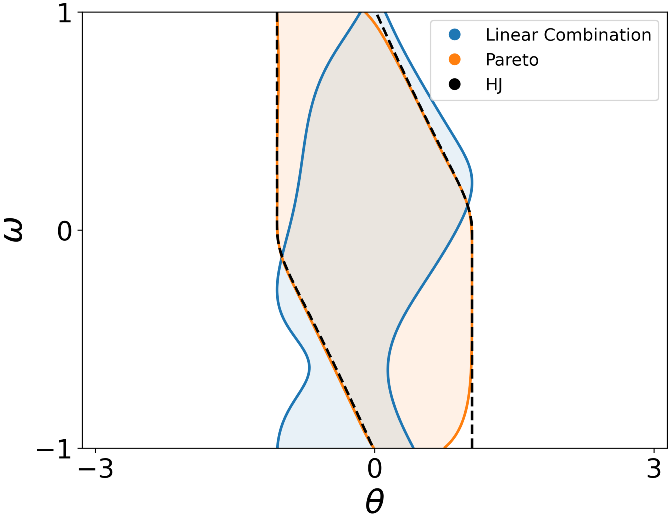
- **Input Space**

$$\mathcal{U} = \{u : -1 \leq u \leq 1\} \tag{46}$$

- **Safe Set**

$$\mathcal{S} = \{(\theta, \omega) : -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}, -1 \leq \omega \leq 1\} \tag{47}$$

Inverted Pendulum



- State Space: 12 dimensions; Input Space: 4 dimensions.
- State Variables: Position (x, y, z); Velocity (v_x, v_y, v_z); Euler Angles (ϕ, θ, ψ); Angular Velocity $(\omega_x, \omega_y, \omega_z)$
- System Dynamics:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{v}} \\ \dot{\phi} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ \frac{1}{m}\mathbf{F} - \mathbf{g} \\ \mathbf{R}\omega \\ \mathbf{I}^{-1}\tau \end{bmatrix}$$

where $\mathbf{R}$ is the rotation matrix, $\mathbf{I}$ is the inertia matrix, and τ is the control torques.

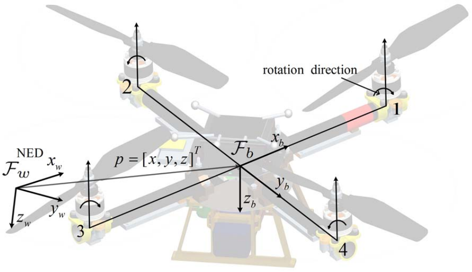
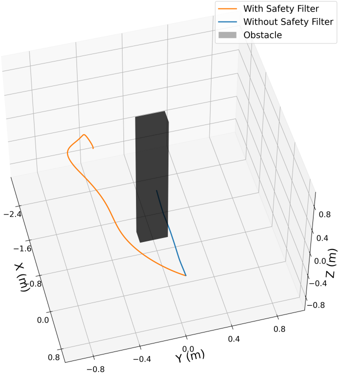


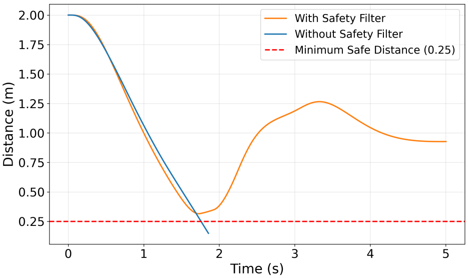
Figure: Quadrotor

Quadrotor

- Obstacle: a rectangle, $-1.25 \le x \le -0.5, -0.125 \le y \le 0.125$
- Safe Set: $x \le -1.5$ or $x \ge -0.75$ or $y \le -0.375$ or $y \ge 0.375$



(a) Trajectory



(b) Distance to obstacle

Pareto Control Barrier Function for Inner Safe Set Maximization Under Input Constraint

Thank you for listening !

Any Questions ?

UC Berkeley